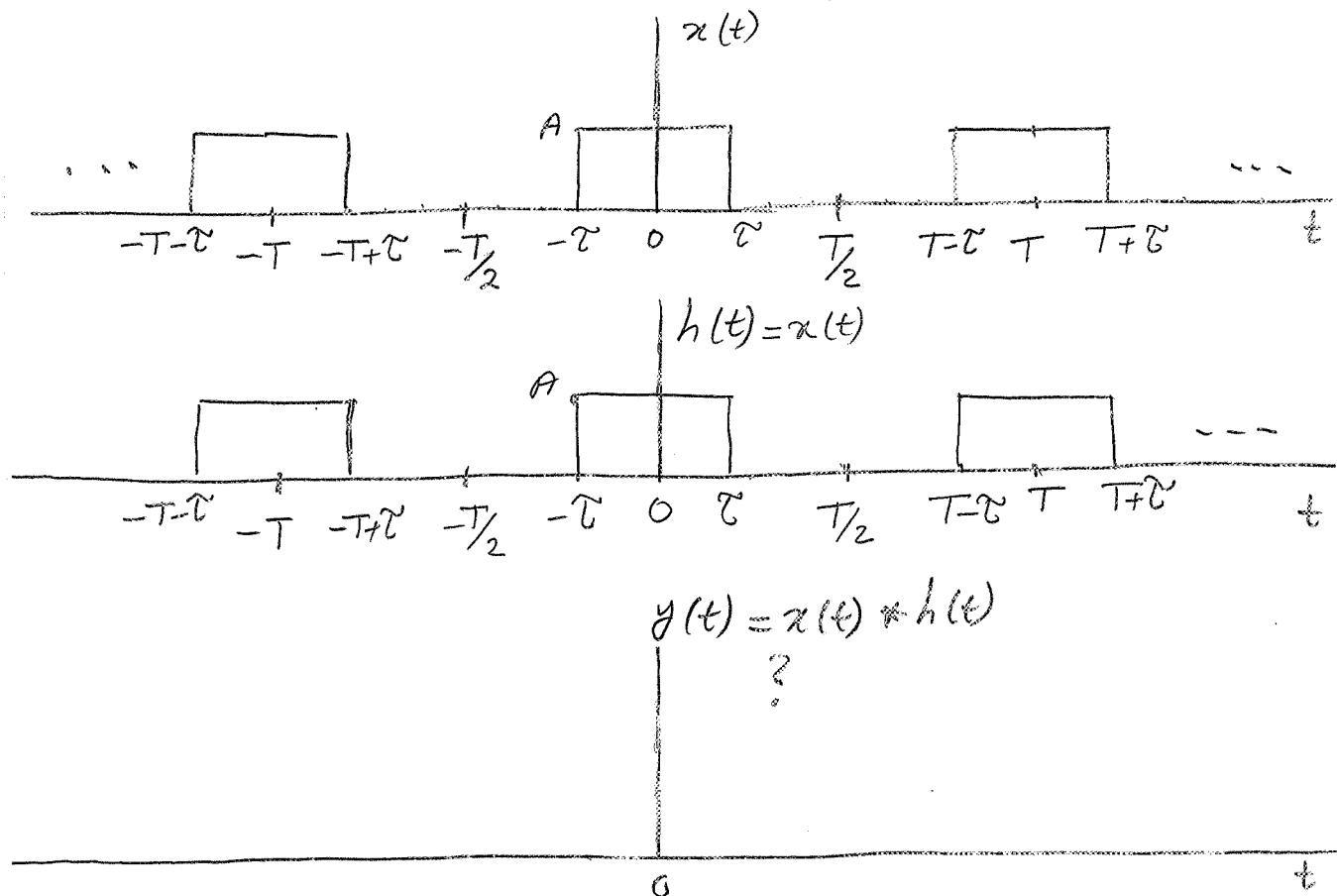


Ex A Find $y(t) = x(t) * h(t)$ where $x(t)$ and $h(t)$ are shown below. [Note that $x(t) = h(t)$].



$$y(t) = \frac{1}{T} \int_{-T/2}^{T/2} x(\beta) h(t-\beta) d\beta = \frac{1}{T} \int_{-T/2}^{T/2} x(\beta) x(t-\beta) d\beta$$

$$x(t) = x(t+T) = \begin{cases} A & -\tau \leq t \leq \tau \\ 0 & \tau < |t| < \frac{T}{2} \end{cases}$$

$$y(t) = x(t) * x(t) \quad \text{since } h(t) = x(t).$$

A different way to compute $y(t)$ using properties of Convolution:

* shift both signals to the right, each by τ .

Thus

$$v(t) = y(t-2\tau) = x(t-\tau) * x(t-\tau) = g(t) * g(t)$$

where $g(t) = x(t-\tau)$.

$$\text{Now } v(t) = \frac{1}{T} \int_0^T g(\alpha) g(t-\alpha) d\alpha$$

$$\text{But } g(t) = A u(t) - A u(t-2\tau)$$

$$\text{and } g(t-\alpha) = A u(t-\alpha) - A u(t-\alpha-2\tau)$$

Case 1 $v(t) = 0$ for $-4\tau < t < 0$

Case 2 For $0 \leq t \leq 4\tau$:

$$v(t) = \frac{A^2}{T} \int_0^T [u(\alpha) - u(\alpha-2\tau)] [u(t-\alpha) - u(t-\alpha-2\tau)] d\alpha$$

$$v(t) = \frac{A^2}{T} \left\{ \int_0^T u(\alpha) u(t-\alpha) d\alpha - \int_0^T u(\alpha) u(t-\alpha-2\tau) d\alpha \right. \\ \left. - \int_0^T u(\alpha-2\tau) u(t-\alpha) d\alpha + \int_0^T u(\alpha-2\tau) u(t-\alpha-2\tau) d\alpha \right\} \quad (I)$$

Now

$$p(t) = \int_0^T \underbrace{u(\alpha)}_{\alpha \geq 0} \underbrace{u(t-\alpha)}_{t \geq \alpha} d\alpha = \int_0^t d\alpha = t u(t)$$

$$p(t) = u(t) * u(t) = t u(t)$$

Second term of (I) is:

$$p(t-2\tau) = u(t) * u(t-2\tau) = (t-2\tau) u(t-2\tau)$$

Third term of (I) is:

$$p(t-2\tau) = u(t-2\tau) * u(t) = (t-2\tau) u(t-2\tau)$$

Fourth term of (I) is:

$$p(t-4\tau) = (t-4\tau) u(t-4\tau)$$

Thus

$$v(t) = \frac{A^2}{T} \left\{ p(t) - 2 p(t-2\tau) + p(t-4\tau) \right\}$$

$$\text{but } v(t) = y(t-2\tau) = x(t-\tau) * x(t-\tau)$$

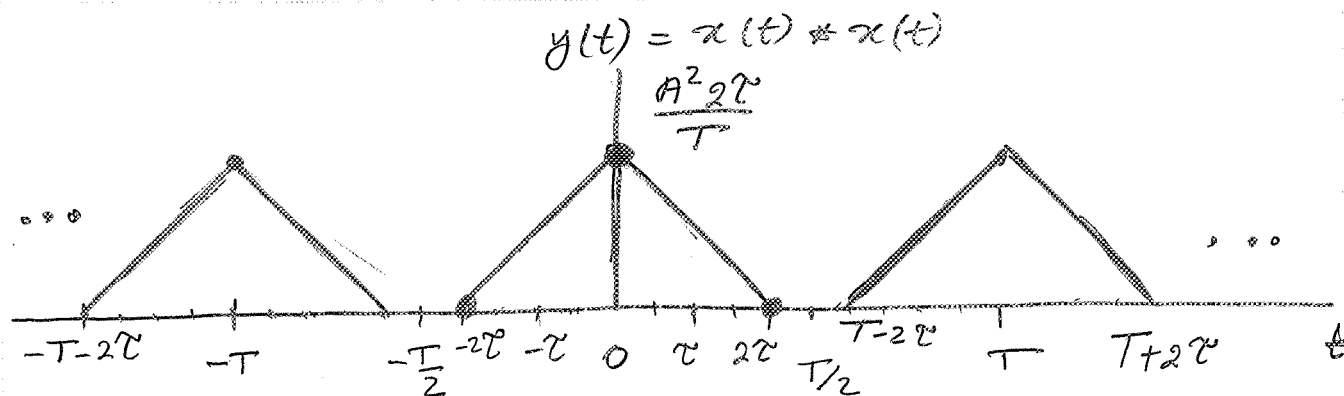
$$\text{and } v(t+2\tau) = y(t) = x(t) * x(t)$$

Thus

$$y(t) = \frac{A^2}{T} \left\{ (t+2\tau) u(t+2\tau) - 2t u(t) + (t-2\tau) u(t-2\tau) \right\}$$

$$\text{for } -\frac{T}{2} \leq t \leq \frac{T}{2}$$

and that $y(t) = y(t+T)$ is periodic.



or

$$y(t+T) = y(t) = \begin{cases} \frac{A^2}{T}(t+2\tau) & -2\tau \leq t \leq 0 \\ \frac{A^2}{T}(-t+2\tau) & 0 \leq t \leq 2\tau \end{cases}$$

Ex B

Find spectrum of $y(t) = y(t+T) = x(t) * h(t)$
- given in Ex A.

Solution

Note that $y(t)$ is shown in the figure above.
Since $y(t)$ is periodic the spectrum or frequency representation of $y(t)$ is obtained using Fourier series as

$$d_n = \frac{1}{2T} \int_{-T/2}^{T/2} y(t) e^{-j\frac{2\pi n}{T}t} dt, \text{ for } -\infty < n < \infty$$

We can compute d_n using the above directly or use the convolution property of the Fourier series as follows.

If: $x(t) = x(t+T) \xleftrightarrow{Fs} C_n$

and $h(t) = h(t+T) \xleftrightarrow{Fs} b_n$

Then

$$y(t) = y(t+T) = x(t) * h(t) \xleftrightarrow{Fs} d_n = C_n \cdot b_n$$

Now, $x(t) = h(t)$ and the Fourier series

Coefficients C_n were computed in Ex. 1, page 3-5 of these notes as

$$C_n = \frac{2AT}{T} \text{Sinc}\left(\frac{2\pi n}{T} \tau\right) \text{ for } -\infty < n < \infty$$

$$C_n = b_n$$

Thus
$$d_n = \left(\frac{2AT}{T}\right)^2 \text{Sinc}^2\left(\frac{2\pi n}{T} \tau\right) = C_n^2$$

for $-\infty < n < \infty$

⊗